Effect of Winkler Foundation on the Dynamic Analysis of Euler-Bernoulli Beam Resting on Elastic Foundation Using Integral Numerical Method

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Abstract
This study examined how a Winkler foundation influences the dynamic analysis of an Euler-Bernoulli beam placed on an elastic foundation by employing an Integral-Numerical method, which simplifies to an ordinary differential equation using a series representation of the Heaviside function. The dynamic responses of the beam, including normalized deflection and bending moment, were analyzed for various velocity ratios under conditions of moving loads and moving masses. In general, a closed-form solution for the generalized mathematical model of a prismatic beam was derived using a symbolic programming technique with MAPLE 18. The findings indicated that the inclusion of an elastic foundation along with adequate reinforcement in beams and beam-like structures decreases vibration intensity, ensures safe load passage, and extends the lifespan of the beam.

Keywords: Axial force, elastic foundation, foundation stiffness, integral-numerical method, moving load.

Introduction
A beam serves as a crucial structural component, serving as a basic and precise framework to evaluate complex engineering structures like turbine and compressor blades, aircraft wings, robotic arms, spacecraft antennas, building frameworks, bridges, and vibrating drills similar to a beam. For over a century, researchers have been interested in the vibration analysis of bridge structures that experience loads or moving masses. This topic gained attention in civil engineering for constructing railways and bridges, as well as in mechanical engineering for crane trolleys that operate on their beams, along with machining applications. The challenge arises from the realization that bridges subjected to the impact of moving vehicles or trains can experience internal dynamic deflections and stress that are substantially higher than those caused by static loads. The examination of the dynamic response of structures like beams and slabs under moving loads has captured the interest of many scholars in engineering, applied physics, and applied mathematics. In this scenario of moving load problems, the mass of the load plays a significant role since its position shifts continuously. Considerable research has focused on this category of dynamic issues when the structural elements have consistent cross-sections.

The method was founded on the generalized Galerkin's technique and integral transformations, with the assumption of a beam featuring a uniform cross-section. In all instances, the focus has been on uniform beams, with non-uniform beams addressed only under classical boundary conditions. Additionally, the work of Oni and Awodola (2011) explored the response of simply supported rectangular plates that carry moving masses while resting on variable Winkler elastic foundations. For solving these equations, a modification of Struble's technique along with the method of integral transformations was utilized. The set of ordinary differential equations was then simplified and resolved using a modified asymptotic staircase method. Although this study is significant, it is restricted to a beam with conventional termination conditions that are simply supported. Other research related to heterogeneous beams includes contributions from Omolofe (2012) and Oni and Ogunyebi (2018). Abdelghany et al. (2015) examined the effects of variations in traveling velocity and the influence of increasing the magnitude of the moving load on the dynamic response of a non-uniform Euler-Bernoulli simply supported beam by employing the Galerkin and Runge-Kutta methods. It is important to note that the majority of investigations in this field have focused solely on classical boundary conditions. Usman (2019) studied the series solution for an Euler-Bernoulli beam subjected to a concentrated load. The findings indicated that as the mass increases, the deflection also rises; conversely, as the mass of the beam decreases, the deflection increases as well. Adedowole et al. (2023) conducted a critical evaluation of a non-Winkler Timoshenko Beam, which included rotary inertial correction and was subjected to paired loads on subgrade. They employed a robust method of Galerkin and integral transforms to derive analytical procedures for the governing equation of motion. The analysis of the vibrating system's solutions was conducted, and various results were presented in the form of plotted curves.

In many practical situations, it is often more feasible to utilize non-classical boundary conditions since ideal boundary conditions are seldom achieved. The issue of bridge-vehicle interaction is a prevalent challenge in the analysis of moving loads and constitutes a significant area of research. If the speed of a vehicle is considerably low, it cannot be classified as a moving load scenario because it behaves like a static load condition at such low speeds. In this case, traditional methods can effectively address the problem. Through mathematical analysis and calculations, these issues can be resolved. Structural vibrations can arise from vehicle movements, earthquakes, river currents, and wind. For safety considerations in design, various factors must be taken into account, including the mass of the body and the moving structure, as well as the inertia resulting from the moving mass of the structure due to eccentric loading.

Bridges are typically categorized into four main types:

(i) beam bridges,
(ii) arch bridges,
(iii) suspension bridges, and
(iv) cantilever bridges.

The primary distinction among these four types of bridges lies in the lengths they can span in a single span. A span refers to the distance between two supporting bridge pillars. Modern girder bridges are capable of spanning distances up to 60 meters, while modern arch bridges can safely manage spans of up to 240 meters. Suspension bridges represent the height of engineering capability, being able to reach spans of up to 2,100 meters. The remarkable ability of an arch bridge to span seven times longer than a beam bridge or seven times more than a suspension bridge is attributed to the way each bridge type manages forces of compression and tension.

Mathematical Formulation

Examine a non-uniform Euler-Bernoulli beam of length L situated on a Winkler foundation, which is subjected to a partially distributed uniformly moving load. The oscillatory dynamics of this system are characterized by the following partial differential equation.

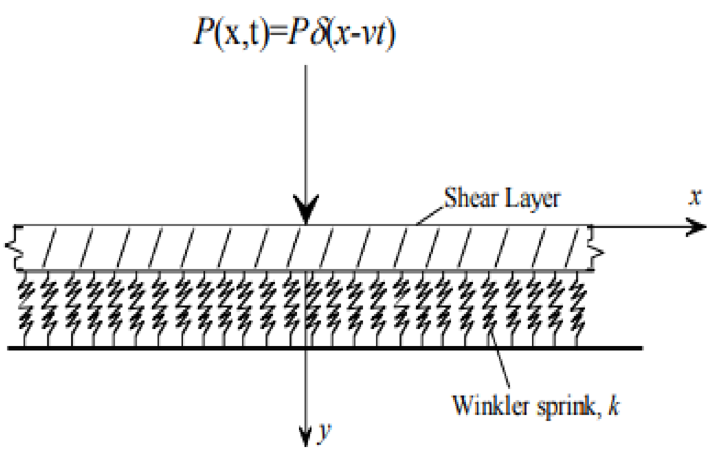

Fig. 1: Beams supported by the Pasternak foundation experience moving loads.
Source: scholarsmine.mst.edu

$$\mathrm{EIL_{xxxx}(x,t) + mL_{tt}(x,t) + cL_t(x,t) - k_1L_{xx}(x,t) = P(x,t) - Q(x,t)} \tag{1}$$

Of which
$\mathrm{L(x,t)}$ is the displacement of the beam

x is the spatial coordinate measured along the length of the beam in m (metre)

t is the time in seconds

EI is the bending stiffness of the beam in Nm^2

m is the linear mass of the beam in Kgm^{-1}

c is the linear viscous damping coefficient of the beam in Nsm^{-2}

k_1 is the flux beam shear number in Pa

$P(x,t)$ are the applied external load per unit length inNm^{-1} , respectively

For the Winkler model, $\mathrm{Q(x,t) = kL(x,t)}$, where k is the elastic constant of the foundation per unit length (N/m^2).

The equation above, can be written as

When the external load $P(x,t)$ is a concentrated load moving at a constant speed v, defined as

$$\mathrm{P(x,t) = \rho\delta(x - vt)} \tag{2}$$

Convective acceleration operator $\mathrm{L_{tt}}$ is given as:

$$\frac{d^2}{dt^2} = L_{tt}(x,t) + 2vL_{tx}(x,t) + v^2L_{xx}(x,t) \tag{3}$$

For simply supported beams has finite length$L$, the possible condition boundary is described Mathematically,

$$\mathrm{L(0,t) = L(L,t) = 0} \tag{4}$$

$$L'(0,t) = L''(L,t) = 0 \tag{5}$$

At the beginning, the beam that is being supported is assumed to be stationary. Consequently, the initial boundary conditions are as follows:

$$\mathrm{L(x,0) = L_{tt}(x,0) = 0} \tag{6}$$

Method of Solution

$$EIL_{xxxx}(x,t) + mL_{tt}(x,t) + cL_t(x,t) - k_1L_{xx}(x,t) + kL(x,t) = \rho\delta(x - vt) \tag{7}$$

$$L_{xxxx}(x,t) = \sum_{n=1}^{\infty} X_n^{iv}(x) T_n(t) \tag{8}$$

$$L_{xx}(x,t) = \sum_{n=1}^{\infty} X_n^{11}(x) T_n(t) \tag{9}$$

$$L(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) \tag{10}$$

$$L_{tt}(x,t) = \sum_{n=1}^{\infty} X_n(x) \ddot{T}_n(t) \tag{11}$$

$$L_t(x,t) = \sum_{n=1}^{\infty} X_n(x) \dot{T}_n(t) \tag{12}$$

$$EI \sum_{n=1}^{\infty} X_n^{iv}(x) T_n(t) + m \sum_{n=1}^{\infty} X_n(x) \ddot{T}_n(t) + c \sum_{n=1}^{\infty} X_n(x) \dot{T}_n(t) - k_1 \sum_{n=1}^{\infty} X_n^{11}(x) T_n(t) + k \sum_{n=1}^{\infty} X_n(x) T_n(t) = \rho\delta(x - vt) \tag{13}$$

For free vibration

$$\sum_{n=1}^{\infty} X_n^{\mathrm{iv}}(x) T_n(t) = \mu\omega_n^2 X_n(x) T_n(t) \tag{14}$$

$$EI\mu\omega_n^2 X_n(x) T_n(t) + m \sum_{n=1}^{\infty} X_n(x) \ddot{T}_n(t) + c \sum_{n=1}^{\infty} X_n(x) \dot{T}_n(t) - k_1 \sum_{n=1}^{\infty} X_n^{11}(x) T_n(t) + k \sum_{n=1}^{\infty} X_n(x) T_n(t) = \rho\delta(x - vt) \tag{15}$$

Integrating over the entire length of the beam and applying the orthogonal property (15), to give

$$\int_0^{a_\alpha} E\, I\mu\omega_n^2 X_n(x) T_n(t) dx + m \int_0^{a_\alpha} \sum_{n=1}^{\infty} X_n(x) \ddot{T}_n(t) dx + c \int_0^{a_\alpha} \sum_{n=1}^{\infty} X_n(x) \dot{T}_n(t) dx - k_1 \int_0^{a_\alpha} \sum_{n=1}^{\infty} X_n^{11}(x) T_n(t) dx + k \int_0^{a_\alpha} \sum_{n=1}^{\infty} X_n(x) T_n(t) dx = \int_0^{a_\alpha} \rho\, \delta(x - vt) dx \tag{16}$$

Multiply both sides of equation (16) by $X_k(x)$

$$\int_0^{a_\alpha} E\, I\mu\omega_n^2 X_n(x) X_k(x) T_n(t) dx + m \int_0^{a_\alpha} \sum_{n=1}^{\infty} X_n(x) X_k(x) \ddot{T}_n(t) dx + c \int_0^{a_\alpha} \sum_{n=1}^{\infty} X_n(x) X_k(x) \dot{T}_n(t) dx - k_1 \int_0^{a_\alpha} \sum_{n=1}^{\infty} X_n^{11}(x) X_k(x) T_n(t) dx + k \int_0^{a_\alpha} \sum_{n=1}^{\infty} X_n(x) X_k(x) T_n(t) dx = \int_0^{a_\alpha} \rho\, \delta(x - vt) X_k(x) dx \tag{17}$$

But

$$\int_0^{a_\alpha} X_n(x) X_k(x) dx = 1 \begin{cases} 0, n \neq k \\ 1, n = k = 1 \end{cases} \tag{18}$$

to obtain

$$EI\mu\omega_n^2 T_n(t) + m\ddot{T}_n(t) + c\dot{T}_n(t) - k_1 \int_0^{a_\alpha} \sum_{n=1}^{\infty} X_n^{11}(x) X_k(x) T_n(t) dx + kT_n(t) = \int_0^{a_\alpha} \rho\, \delta(x - vt) X_k(x) dx \tag{19}$$

Let

$$X_n(x) = \sin \frac{n\pi x}{a_\alpha} \tag{20}$$

$$X_n'(x) = \frac{n\pi}{a_\alpha} \cos \frac{n\pi x}{a_\alpha} \tag{21}$$

$$X_n''(x) = (-)\frac{n^2\pi^2}{{a_\alpha}^2}\sin\frac{n\pi x}{a_\alpha} \tag{22}$$

$$X_k(x) = \sin\frac{k\pi x}{a_\alpha} \tag{23}$$

Substituting equation (20)-(23) into equation (19), to obtain

$$EI\mu\omega_n^2 T_n(t) + m\ddot{T}_n(t) + c\dot{T}_n(t) + k_1\int_0^{a_\alpha}\frac{n^2\pi^2}{{a_\alpha}^2}\sin\frac{n\pi x}{a_\alpha}\sin\frac{k\pi x}{a_\alpha}T_n(t)dx$$
$$+kT_n(t) = \int_0^{a_\alpha}\rho\,\delta(x - vt)\sin\frac{k\pi x}{a_\alpha}dx \tag{24}$$

Collect the like term in (24), one obtain

$$m\ddot{T}_n(t) + c\dot{T}_n(t) + EI\mu\omega_n^2 T_n(t) + kT_n(t) + k_1\int_0^{a_\alpha}\frac{n^2\pi^2}{a_\alpha}\sin\frac{n\pi x}{a_\alpha}\sin\frac{k\pi x}{a_\alpha}T_n(t)dx$$
$$= \int_0^{a_\alpha}\rho\,\delta(x - vt)\sin\frac{k\pi x}{a_\alpha}dx \tag{25}$$

Result and Discussions

The equation governing the motion of the beam is a non-homogeneous partial differential equation of fourth order that has varying coefficients and heterogeneity. The dynamic beam problem's solution is derived using an integral numerical method, which reflects the displacement response of the beam. Consequently, a numerical representation of the findings from this analysis is provided. The graphs indicate that as the axial force values of the fixed foundation increase, the deflection magnitude correspondingly decreases, particularly for the foundation's stiffness. Furthermore, for varying fixed axial force values associated with foundation stiffness, the horizontal deflection of the beam diminishes as the foundation stiffness increases. Thus, heightened foundation stiffness ensures enhanced stability and reliability in the structural design.

Table 1: Deflection against Time(s) at $E = 5, E = 10, E = 15$

TIME(s)	$E = 5Nm^{-2}$	$E = 10Nm^{-2}$	$E = 15Nm^{-2}$
0	0	0	0
1	0.00259705	0.00167358	0.000153502
2	0.00311542	0.000832119	0.000355707
3	0.00093947	0.00s0750946	0.000557563
4	0.00180220	0.00146950	0.000723602
5	0.00288728	0.000562520	0.000834445
6	0.00164498	0.00126111	0.000885838
7	0.00156598	0.000979008	0.000884987
8	0.00251070	0.000852127	0.000846539
9	0.00203035	0.00122698	0.000787629
10	0.00161944	0.000815122	0.00072427

From the table 1, we have a graph of the beam's dynamic response at different values, i.e E = 5,10,15 respectively.

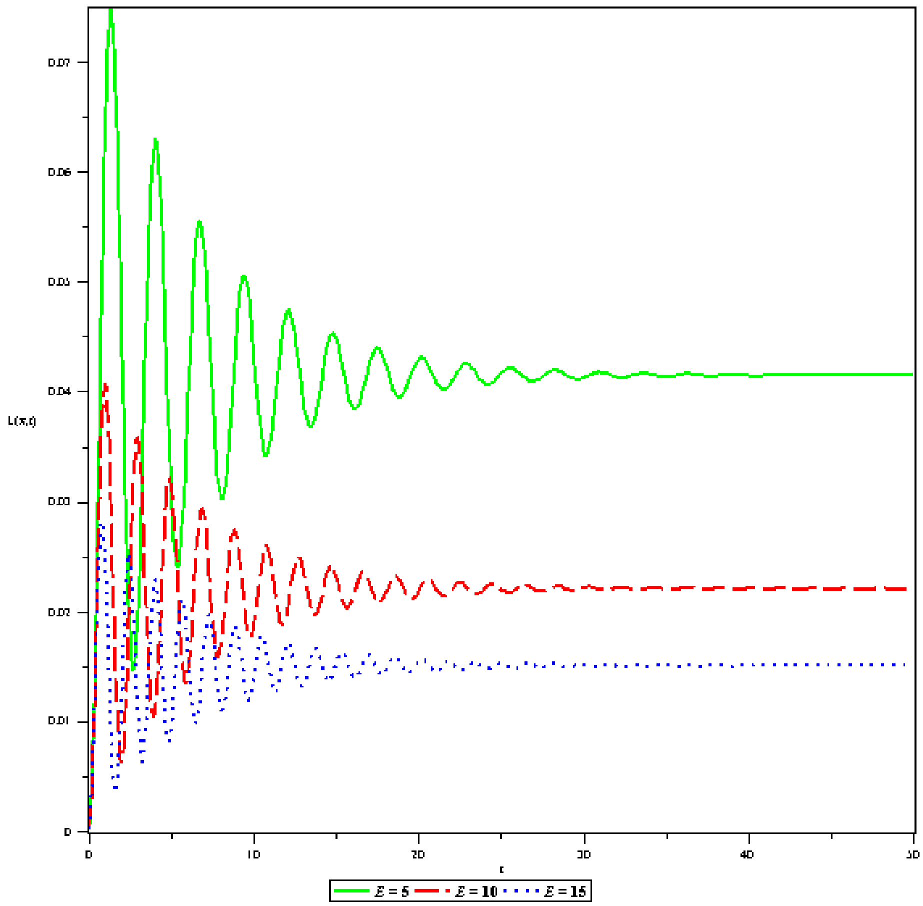

Fig. 2: Dynamic response of the beam at different values of E for $m = 3kgm^{-1}, \mu = 1kgm^{-1}, I = 3kgm^2, \omega_n^2 = 1rads^{-1}, L = 1m, c = 1Nsm^{-2}, g = 10ms^{-2}, w_1 = 0.1, k = 1pa$

Table 2: Deflection against Time(s) at $E = 5, E = 10, E = 15$.

TIME(s)	$E = 5Nm^{-2}$	$E = 10Nm^{-2}$	$E = 15Nm^{-2}$
0	0	0	0
1	1.99597	0.715517	0.167837
2	0.0109109	0.808465	0.503997
3	1.91324	-0.0020497	0.660954
4	0.0847565	0.593629	0.474654
5	1.77213	0.873977	0.132594
6	0.216283	0.873977	-0.018130
7	1.58111	0.467881	0.173828
8	0.396818	0.921680	0.509228
9	1.35198	0.0718092	0.642168
10	0.614450	0.345256	0.434293

From the table 2, we have a graphs of the beam's dynamic response at different values of E=5,10,15 respectively.

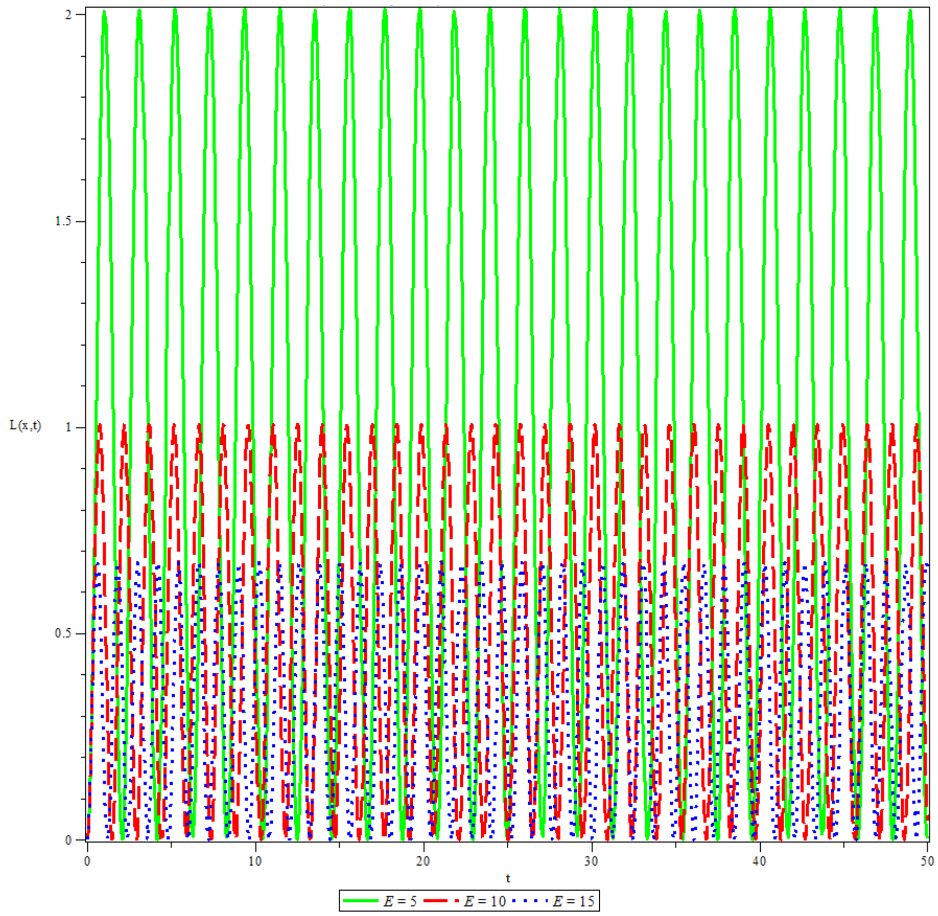

Fig. 3: Dynamic response of the beam at different values of E for $m = 3kgm^{-1}, \mu = 1kgm^{-1}, I = 3kgm^2, \omega_n^2 = 1rads^{-1}, L = 1m, c = 1Nsm^{-2}, g = 10ms^{-2}, w_1 = 0.1rads^{-1}, k = 1pa$

Table 3: Deflection against Time(s) at $E = 5, E = 10, E = 15$

TIME(s)	$E = 5Nm^{-2}$	$E = 10Nm^{-2}$	$E = 15Nm^{-2}$
0	0	0	0
1	0.000153502	0.00167358	0.00259705
2	0.000355707	0.000832119	0.00311542
3	0.000557563	0.000750946	0.00093947
4	0.000723602	0.00146950	0.00180220
5	0.000834445	0.000562520	0.00288728
6	0.000885838	0.00126111	0.00164498
7	0.000884987	0.000979008	0.00156598
8	0.000846539	0.000852127	0.00251070
9	0.000787629	0.00122698	0.00203035
10	0.000724276	0.000815122	0.00161944

From the table 3, we have a plot of graphs dynamic response of the beam at various values of $E = 5, E = 10, E = 15$ respectively.

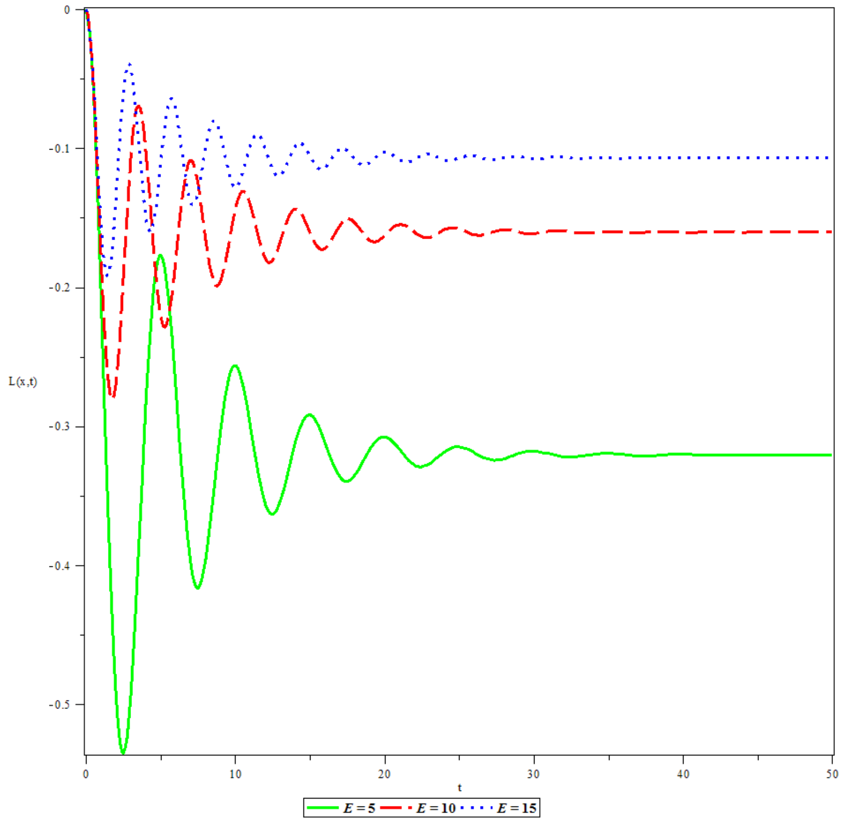

Fig. 4: The response of the beam to the moving load at various values of E for $m = 3\text{kgm}^{-1}, v = 3.3ms^{-1}, \mu = 1\text{kgm}^{-1}, I = 1kgm^2, L = 1m, c = 1Nsm^{-2}, g = 10ms^{-2}, w_1 = 0.1rads^{-1}, \omega_n^2 = 1rads^{-1}$

Conclusion

The vibration characteristics of beams supported by an elastic foundation under moving loads are investigated. The findings clearly demonstrate that having an elastic foundation and adequately reinforcing beams and beam-like structures lessens vibration intensity, ensuring safe transit of solid objects (loads) and extending the lifespan of the beam structure. Additionally, it is important to highlight that this outcome can serve as a foundational engineering design criterion. The graph illustrates the deflection profile of non-uniform beams with various levels of foundation stiffness. Figure 2 shows the active retort of the beam at diverse morals of $E = 5,10,15$ respectively. It is observed that the dynamic response of the beam increases as the length of the load increases.

Figure 3 displays the dynamic reply of the beam at diverse values $E = 5,10,15$, it is pragmatic that the dynamic comeback of the beam increases as the length of the load increases.

Figure 4 displays the dynamic retort of the beam at unlike values of $E = 5,10,15$, respectively. It is perceived that the vibrant response of the beam increases as the dimension of the load increases.

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